DOCUMENT RESUME

ED 420 523 SE 061 562

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TITLE Epistemological Messages Conveyed by High School and College Mathematics Textbooks.
PUB DATE 1998-04-00
NOTE 21p.; Paper presented at the Annual Meeting of the American Educational Research Association (San Diego, CA, April 13-17, 1998).
PUB TYPE Reports - Research (143) -- Speeches/Meeting Papers (150)
EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS *Calculus; *Definitions; High Schools; Higher Education; *Mathematics Education; *Textbook Content; *Textbook Evaluation

ABSTRACT

Mathematics textbooks embody a particular set of assumptions about mathematics or the mathematics intended for students at a particular level. An epistemological analysis of textbooks can provide some context for understanding, for example the difficulties many students encounter when moving from high school to collegiate mathematics. In this study, it is considered how typical precalculus, calculus, and analysis texts treat the topic of continuity. It is found that these texts send conflicting messages about the purpose and use of mathematical definitions. (Author)

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Epistemological messages conveyed by high school and college mathematics textbooks

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Prepared for the American Educational Research Association Annual Meeting, San Diego, April 13-17 1998

Warning: This is a working draft.
Treat appropriately.

Mathematics textbooks embody a particular set of assumptions about mathematics or the mathematics intended for students at a particular level. Thus, an epistemological analysis of textbooks can provide some context for understanding, for example, the difficulties many students encounter when moving from high school to collegiate mathematics. In this study, we consider how a typical precalculus, calculus, and analysis text treat the topic of continuity. We find that these texts send conflicting messages about the purpose and use of mathematical definitions.

MOTIVATION FOR STUDY

Many American students have difficulty making a transition from high school to college level mathematics. This difficulty can be traced, at least in part, to students' beliefs about what mathematics is. Several studies have indicated how students' beliefs can conflict with the beliefs needed to succeed at a particular level (Schoenfeld, 1989; Schommer, et al., 1992; Tall, 1992). The focus of this study is on one possible source for conflicting beliefs—the messages sent by high school and college level mathematics textbooks— which, for better or for worse, tend to have a strong influence on the way mathematics is taught and learned.

In this paper, I will look at precalculus, calculus, and analysis texts, which span the space between the high school and college curricula. The presentation of the mathematics in texts at these levels is very similar. Each chapter starts with definitions, followed by explanation, interspersed with examples. Then come some theorems, followed by some proof. Then come some questions for the students to do on their own.

However, when we look closely at the content of the texts, we find very different epistemological assumptions, which I believe help explain why many students are poorly prepared for advanced mathematical study.

THE FOCUS TOPIC: CONTINUITY

I have chosen to look at the presentation of one particular topic—continuity. Continuity, like most mathematical notions, can be characterized both informally and formally. Informally, it can be characterized by tracing a graph without lifting the pencil. Formally, it can be defined in terms of limits or mappings of open sets.

While continuity is not always treated in the precalculus curriculum[1], there are several features that make it a good topic for study. First, students have a notoriously difficult time understanding the formal definition, which is usually presented in calculus or analysis (Nadler, 1994; Tall & Vinner, 1981). Second, because there is a significant difference in the difficulty of the informal and formal notions, it is easier for us to see differences in treatment at the three curricular levels. And even though continuity is not always treated in precalculus, the assumptions underlying the precalculus treatment that we will examine are fairly representative of messages at that level.

THE TEXTBOOKS IN THIS STUDY

I have chosen three textbooks for my study, which I will call *Precalculus*, *Calculus*, and *Analysis*. I used two criteria for choosing the texts. One, that it be a popular text whose epistemological assumptions are representative of the most widely used texts at that curricular level. Two, that the topic of continuity is treated, so I have some consistency for my analysis. I am deliberately omitting references because my purpose in this paper is not to critique these particular texts. Rather, my purpose is to illustrate the types of conflicts that may arise from clashing epistemological messages sent by books like these.

EPISTEMOLOGICAL MESSAGES FROM A PRECALCULUS TEXT

What type of definition is given?

[1] In fact, of the 27 precalculus texts I reviewed, only 9 treated continuity at all and only 3 of those provided more than an informal characterization of the property.

1. Definition is informal

As the authors claim in the preface, a goal at this level is "to lay an intuitive foundation for calculus." What that means in the case of continuity is to provide an informal definition and expect students to invoke it only to classify certain functions that they are already familiar with.

Here is the way continuity on an interval is defined:

> Continuous Function over an Interval J
>
> A function *f* is continuous on the interval *J* if for all *a* and *b* in *J*, it is possible to trace the graph of the function between *a* and *b* without lifting the pencil from the paper. If *f* fails to be continuous on an interval *J*, then it is discontinuous on interval *J*.

Below we will discuss implications of using an informal definition of continuity.

2. Informal definition is written as if it were formal

One issue here is that the authors do not indicate to students that this definition is informal. It is written as a formal definition (included in a box) and has the language of a definition. It does lack the title "Definition" used for precise mathematical definitions in this text, but I suspect that most students would not pick up on that subtlety.

This example is one of several that we will see at this level that may contribute to confusing messages about the appropriate use of informal and formal reasoning.

3. Treatment assumes familiarity with continuity

It seems reasonable to base an informal definition on familiar notions. However, by not making explicit that connection, students who want to try to make sense of a description like this one may get confused. We see an example of a confusing message in this text where the authors give an informal description of continuity.

Students are introduced to the notion of continuity by graphing on a graphing utility several functions in a particular viewing rectangle. The text then reads:

> Which of the functions in this Exploration could be sketched by drawing one continuous curve? These are called continuous functions. Which ones

> required you to lift your pencil from the paper at least once? These are discontinuous functions.

Notice the circularity: A continuous function is one that can be drawn by a continuous curve. The authors seem to rely on the fact that the students will already have an understanding of what continuity is (but perhaps haven't applied that idea to functions.)

Also notice that rather than pointing to the complementarity of continuous and discontinuous functions, the authors use unrelated descriptions for each. It seems the authors want to try to describe continuity differently from the "definition" (in the box above), but have difficulty describing such an intuitive notion in noncircular terms.

How are students supposed to use the definition?

1. *Questions involve classification*

The informal definition of continuity limits the types of questions that can be asked. As expected, all of the questions at the end of the section ask students to classify functions as continuous or discontinuous. There are two types of problems. There are eight questions of the first type, that have the following directions:

> Assume the graph is complete. Identify the function as continuous or discontinuous. If it is discontinuous, name two intervals on which it is continuous and two intervals on which it is discontinuous.

There are six questions of the second type that have the following directions:

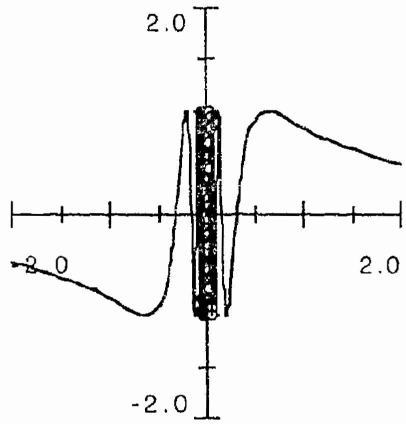
> Determine the points of continuity of each function.

It is not clear from the directions how students are supposed to determine the points of continuity, but based on text examples, it seems the authors expect students to use a graphing calculator and draw inferences from the graph. This ambiguity in directions, by the way, may send a message to students to follow worked examples rather than try to reason through textbook problems on their own. And if students make it to *Analysis*, where worked examples are few and far between, they may be lost.

2. *Questions (only) ask students to reason from the graphs*

All 14 questions above ask students to determine the continuity of a function from its graph. These questions seem consistent with the goals at this level to give students a rough idea of the notion. But, again, the authors do not explicitly tell students that this reasoning is informal and limited. Thus students may not realize the limitations of reasoning from a graph.

One limitation is that some functions students will encounter later on have misleading graphs. Consider for example $f(x) = \sin(1/x)$, the graph of which looks roughly like:



Because the osciliations get infinitely close around the origin, it is impossible to tell from the graph that the function is undefined, and hence (according to the definitions in both *Precalculus* and *Calculus*) discontinuous at $x=0$. In this case, an algebraic representation of the function is needed.

Another example is:

$$g(x) = \begin{cases} \sqrt{x} & \text{when } x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

The graph of $g(x)$ would look just like the graph of $y = \sqrt{x}$ (since points on the x axis would be hidden by the axis itself.) So how should students know, just from looking at the graph what the algebraic representation of that function is?

In *Precalculus* (and for the most part in *Calculus*) the assumption is that the graphs faithfully correspond to a type of function the students are familiar with. And this seems like a reasonable assumption at this level. Graphs are often useful for understanding the behavior of a function, and are often easier for students to analyze than algebraic formulas.

However, in higher levels of mathematics, students will see functions, like $f(x)$ and $g(x)$, whose graphs are misleading. In fact, part of the motivation for a formal definition of continuity was the existence of monster functions like $f(x)$ and $g(x)$. If the limitations of graphing reasoning aren't made clear, either here or at a later level, students may fail to see these limitations when they should.

3. Parts of tasks are artificial

Note that in the first type of question, in 1 above, students are asked to write an answer in interval notation, which doesn't have anything to do with continuity. It seems the authors are using the context of this identification question as an opportunity to review an algebra skill. The setting is artificial, perhaps sending the message that the motivation for skills comes from textbook authors, not from the mathematics itself.

This message is repeated in problems 15-18, which are ostensibly related to the Intermediate Value Theorem. Students are given a fixed number N and asked to find a value of c such that $f(c) = N$. I do not think students will understand anything about how continuity relates to the Intermediate Value Theorem based on these questions.

4. Worked solutions confound syntax and semantics

For each group of questions, there is a prototype worked out in the text. We can look at a worked example to see the type of reasoning the authors expect the students to use. This is the second example about continuity, labeled "Finding continuity in a step function." Note that INT(x) is the greatest integer function, often denoted $\lfloor x \rfloor$.

> Determine the length of the longest interval on which the function $f(x)$ = INT(x) is continuous.
>
> Solution (See Fig 3.15) The longest interval on which f is continuous is 1 unit long. Here's why. Read from the graph that $f(2)\neq 1$ but that $f(2) = 2$. In general, if x is in the interval $[n, n+1)$, where n is an integer, then $f(x) = n$. So f is continuous on the interval $[n, n+1)$ for each n.
> Any interval that contains an integer as an interior point includes a break point for the graph, and f is not continuous on such an interval.

Notice that in the solution, the authors do not explicitly invoke the definition of continuity that they gave. If they did, the solution would look something like:

> The longest interval on which the function is continuous is the longest interval we can trace it without lifting our pencil. When we graph the function, we find that after 1 unit, we must lift our pencil.

Instead of this sort of informal reasoning using the informal definition, the authors use sort of pseudo-algebraic reasoning which makes no use of the given definition. In fact, they do not make clear why the fact that $f(x) = n$ (in line 6) has anything to do with the function being discontinuous. The roles of informal descriptions and definitions are confounded here, which I think makes it difficult for students to understand the role of definitions in mathematics.

5. *There is little motivation for the concept*

It seems reasonable that the first time students are introduced to continuity as a mathematical concept that they are asked to use the notion in a basic way. And classification based on continuity seems like a reasonable task for students at this level. But with this text, students are asked to classify without knowing why continuity is an important topic. There is neither practical nor historical motivation for the topic, and except for a very superficial discussion of the Intermediate Value Theorem for which there are no corresponding questions relating to continuity, no real applications. So precalculus students may wonder why they are classifying functions as continuous and discontinuous.

As a comparison, in Swedish high school textbooks, the first time students see the topic of continuity they are given historical motivation and examples of situations from real life that involve continuous and discontinuous functions (Jacobsson, Wallin, & Wiklund, 1995). They see that dropping a ball can be modeled with a continuous function while postage rates can be modeled with discontinuous functions. Students at this level are not asked to do much with continuity, but spiral back to it in later courses.

<u>What messages are sent at this level?</u>

Based on the treatment of continuity in *Precalculus*, we can get a sense for some of the messages sent to students at this level which may conflict with messages at later levels. First, we see that this treatment confounds formal and informal reasoning. The limitations of informal definitions are not clear, and solutions appear more formal than they are. This may make it difficult for students to learn when informal and formal reasoning is appropriate, which will cause problems especially in *Analysis*.

Second, we see that there is little motivation for the concept. Students may see that one can classify functions using the informal definition. But at this level they may not see why that classification is interesting or why continuity might be an important topic to pay attention to later on. There is no practical or historical motivation nor previewing of topics to come.

So when students see a formal definition of continuity in *Calculus*, they may not understand why a formal definition is warranted.

In short, precalculus students may not recognize the limitations of informal reasoning nor how it is related to, but different from, formal reasoning. Thus, they may not be in a position to either appreciate formal treatments of mathematics that come in later courses or connect formal treatments to their informal understandings.

EPISTEMOLOGICAL MESSAGES FROM A CALCULUS TEXT

What type of definition is given?

1. *Definition is formal*

In *Calculus*, students are expected to apply definitions and theorems, though at this stage they aren't really expected to know why definitions are important and what they really mean. In the case of continuity, this means that the students are asked to satisfy the definition of continuity (at a point and in an interval). As a result, the authors provide formal definitions:

(1) Definition A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

(2) Definition A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a number a if

$$\lim_{x \to a^-} f(x) = f(a)$$

(3) Definition A function f is continuous on an interval if it is continuous at every number in the interval. (At an endpoint of the interval we understand continuous to mean continuous from the right or continuous from the left.)

Limits are defined earlier in the book in terms of epsilons and deltas, but students are not required to use the formal definition of limit in any of the problems. This definition is precise, unlike the informal definition in *Precalculus*, so it can be satisfied. However, as we see when we examine the questions, there is so much emphasis on satisfying the definition in *Calculus*, that the informal notion is all but lost.

2. *Informal definition mentioned, but not used*

There is an attempt to link the formal definition to an informal one. After the first definition, the text reads, "Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it. The graph can be drawn without removing your pen from the paper." We see here that there is a clear distinction between the formal and informal notions. The "definition" of continuity from *Precalculus* is considered an interpretation of the more formal definitions given in *Calculus*.

However, as we will see in more detail below, it is interesting that this informal characterization is not expected to be used by students in most problems. It seems the informal characterization here is intended to give students some sense of what continuity means, though they aren't expected to use that knowledge.

3. *Definitions are not motivated*

While there is some context provided for Definition 1, there is no motivation at all for Definitions 2 and 3. It is not made clear to students why three definitions are needed and why they are formulated in this way.

There is some explanation of why the definition makes sense: "Intuitively, f is continuous at a if $f(x)$ gets closer and closer to $f(a)$ as x gets closer and closer to a." But it is significant that this intuitive description, like all the informal characterizations given here, is written as an explanation rather than a motivation.

Another reason why the definitions here seem unmotivated is that most questions ask students to classify functions for which only an informal definition is needed. There are 2 functions like $g(x)$ from above which are included at the end of the list of problems. But I suspect that these functions would have little chance of changing the messages sent by the 50 preceding ones. Thus, I suspect many students at this level would not be able to see why the formalism affords them anything more than busywork.

How are students supposed to use the definition?

1. *Most problems involve satisfying definitions*

Of the 60 questions at the end of the section, 50 of them require students to apply either a definition or a theorem relating to continuity. There are two problems at the beginning similar to the *Precalculus* questions that ask students to reason from a graph. And there are eight questions at the end that are not like any of the prototype questions (most involve an application of the Intermediate Value Theorem.) But by and large, the bulk of the questions in this section are fairly straightforward applications of definitions and limit laws that are modeled by worked examples in the text.

For instance, the instructions for questions 3-11 ask:

> Use the definition of continuity and the properties of limits to show that each function is continuous at the given number (or on the given interval.)

We see a big shift between *Precalculus* and *Calculus* in terms of the role of intuition and rigor. In *Precalculus*, where students were asked mostly to classify functions as continuous or discontinuous, the students were expected to use an intuitive idea of continuity. Here students are mostly asked to satisfy a formal definition without using any intuitive idea of continuity.

2. *Solutions require syntax, not semantics*

The questions require only a syntactic understanding of the definition. We can again look to a worked example to see the type of reasoning expected of students:

> Show that the function $1-\sqrt{1-x^2}$ is continuous on the interval [-1,1].
>
> Solution: If $-1 < a < 1$, then using the Limit Laws, we have
>
> $\lim_{x\to a} f(x) = \lim_{x\to a}(1-\sqrt{1-x^2})$
>
> $= 1 - \lim_{x\to a}\sqrt{1-x^2}$ (by Laws 2 and 7)
>
> $= 1 - \sqrt{\lim_{x\to a}(1-x^2)}$ (by 11)
>
> $= 1 - \sqrt{1-a^2}$ (by 2, 7, and 9)
>
> $= f(a)$

9 Thus by Definition 1, f is continuous at a if $-1 < a < 1$. We must
10 also calculate the right-hand limit at -1 and the left-hand
11 limit at 1.

12 $$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (1 - \sqrt{1 - x^2})$$
13 $$= 1 - \lim_{x \to -1^+} \sqrt{1 - x^2} \quad \text{(as above)}$$
14 $$= 1 - \sqrt{1 - 1^2}$$
15 $$= 1 = f(-1)$$

16 So f is continuous from the right at -1. Similarly
17 $$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (1 - \sqrt{1 - x^2})$$
18 $$= 1 - \lim_{x \to 1^-} \sqrt{1 - x^2} = 1 - 0 = 1 = f(1)$$

19 So f is continuous from the left at 1. Therefore, according to
20 Definition 3, f is continuous on $[-1,1]$.

21 The graph of f is sketched in Figure 2. It is the lower half of
22 the circle $x^2 + (y-1)^2 = 1$.

Note that the argument is lines 3-7 is essentially repeated in lines 11-15 and 16-20. The only difference in the latter cases is that they are taking left and right limits (to check continuity at the endpoints.) So about half of the text in the solution has to do with small details of the situation and not about properties of continuity. This solution does not seem to provoke the need for any semantical notion of continuity.

3. *Results precede motivation*

Part of the etiquette in formal mathematics, which influences the presentation in this text, is that results are often presented before motivation. We saw an example above where the informal characterization for continuity came after the formal definition. And here we see that the motivation for the solution comes after the solution. Line 9 describes the purpose of line 4-8, lines 19-20 describes the purpose of lines 3-18.

This ordering is opposite from that of *Precalculus* arguments. In *Precalculus*, students were first asked to think about differences between different types of functions before they saw the informal definition. I think that the ordering of the *Calculus* material makes it more difficult for students to see the (limited) type of reasoning that is expected at this level.

It isn't clear from the text that the goal is simply to satisfy a formal definition rather than to have a semantical understanding of continuity.

4. *Role of graph is different than in Precalculus*

In *Precalculus*, the graph of the function was used to determine the continuity of a function. In *Calculus*, as the example above illustrates, the graph is related to, but not an essential part of, the reasoning. The solution above is entirely algebraic. The graph of the function comes at the very end (lines 21-22). In *Precalculus*, the fact that this function is the lower half of a circle is importa[illegible] here it is not.

This message is reiterated in the set of questions at the end of the section. In questions 12-18 and 31-35, students are asked to find points of discontinuity and then sketch the graph of the function. The instructions again imply that the graphs come as an afterthought. At best, students may use it to check their work. But, most likely, students will see graphing as yet another tedious part of a question that has nothing to do with a semantical understanding of the problem situation.

5. *Treatment doesn't emphasize need for formal definition*

Most of the problems explicitly require students to use the definition of continuity to analyze functions whose points of discontinuity are obvious. Even functions like

$$f(x) = \begin{cases} 0 & \text{if x is irrational} \\ 1 & \text{if x is rational} \end{cases},$$

which are included at the end, have obvious points of discontinuity. If I tried to graph this function, I would have to lift my pencil at every point. Moreover, the syntactical solutions do not clarify the meaning of continuity.

The types of questions that really get at the need for a formal definition are semantical questions that involve finding entailments of the notion, like we will find in *Analysis*. There aren't really problems like this in *Calculus*, although a few problems at the end at least illustrate the usefulness of theorems about continuity.

I will include one example because it turns out that this question is also asked in *Analysis*. By comparing how the questions are asked we can see the types of reasoning required of students at each level.

> Problem 60. A fixed point of a function f is a number c in its domain such that $f(c) = c$. (The function doesn't move c; it stays fixed.)

(a) Sketch the graph of a continuous function with domain [0,1], whose range is also in [0,1]. Locate a fixed point of f.

This question just helps students apply the definition of fixed point.

(b) Try to draw the graph of a continuous function with domain [0,1] and range in [0,1] that does not have a fixed point. What is the obstacle?

This question helps students see why there is a problem. It turns out the problem has to do with continuity. The function must cross the line $y = x$.

(c) Use the Intermediate Value Theorem to prove that any continuous function with domain [0,1] and range in [0,1] must have a fixed point.

The Intermediate Value Theorem helps prove the claim that function must cross the line $y = x$. Although students are told to use this particular theorem, which hasn't really been proven, they see here a real application (as opposed to an artificial one) of continuity.

At this level, the Intermediate Value Theorem isn't proven, so students may still not have a complete understanding of why the functions described above may have a fixed point. However this type of problem may at least provide a good application for the theorem.

I should point out again that there are very few problems like these in *Calculus*, and they all come at the end of a long list of problems. And in practice, few of these types of problems are assigned or tested. So it is probably still safe to say that the notion of continuity, at this level, is still largely unmotivated.

What messages are sent at this level?

Again we can look at how the messages here mesh with messages at other levels. First, we see, as with *Precalculus*, that there is little motivation for the formal definition. The definitions come more or less out of the blue. The questions do little to help motivate the need for the formal definition since most of them could be answered with an informal one. It appears, then, that neither *Precalculus* nor *Calculus* have prepared students to shift between using informal and formal definitions.

Second, the type of reasoning required here is significantly different from the reasoning in *Precalculus* (and also from *Analysis*) Here the students are mostly asked to satisfy definitions. Most of the questions are entirely syntactic and do not teach students anything new about continuity. Students are not expected to use graphs to guide their reasoning. The message students may get is that the purpose of the formal definition is to make an easy task unnecessarily cumbersome.

Further, the vast majority of problems do not use the analytical power of formal definition.

In short, *Calculus* students may not see the power of formal definitions. Thus they may not appreciate the need to abandon informal reasoning and may see mathematical formalism as foreign and impractical.

EPISTEMOLOGICAL MESSAGES FROM AN ANALYSIS TEXT

<u>What type of definition is given?</u>

1. *Definition is formal*

In *Analysis*, students are expected to invoke a formal definition and use it to find new (to the student) properties of the concept. The definition at this level is more formal than the *Calculus* one, in that it does not explicitly rely on the definition of limit and is couched in general terms:

> Definition Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y. Then f is said to be continuous at p if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that
>
> $$d_Y(f(x), f(p)) < \varepsilon$$
>
> for all points for which $d_X(x, p) < \delta$.
>
> If f is continuous at every point of E then f is said to be continuous on E.
>
> It should be noted that f has to be defined at the point p in order to be continuous at p.

In lines 5 and 6, $d(x,p)$ means the distance between x and p, where d is the metric of the space (on the real line, the metric would be absolute value). Lines 4-6 are a formal way of saying $\lim_{x \to p} f(x) = f(p)$, which is pointed out to students in the *Analysis* text in the form of a theorem which follows this definition.

We will not focus on the technical aspects of this definition, but rather the messages sent by couching the definition this way.

2. Setting for the definition is abstract

In *Analysis*, continuity is defined for functions on an arbitrary metric space instead of on the real line. The author explains in the introduction to the chapter, "The theorems we shall discuss in this general setting would not become any easier if we restricted ourselves to real functions, for instance, and it actually simplifies and clarifies the picture to discard unnecessary hypotheses and to state and prove theorems in an appropriately general context."

This statement gives us insight into the types of understandings expected of students at this level, and how those contrast with the expectations in *Precalculus* and *Calculus*. In earlier courses, continuity was treated in the context of functions for which an intuitive sense of continuity was fairly useful. Here continuity is treated generally, and it may not be clear to students the extent to which their understanding of real valued functions is relevant. The concern is that students may abandon an intuitive sense of continuity that could be useful for guiding their reasoning.

3. Treatment includes little explanation or motivation

While this particular text provides unusually little motivation, its treatment provides a reasonable caricature of texts at this level. Students see Definition 4.1, Theorem 4.2, Corollary, Definition 4.3, Theorem 4.4 and so on with little text in between. The message sent to students is: here's a definition, defined as such by great mathematicians for reasons which you will not know, and you are left pretty much on your own to make sense of it.

How well are students prepared to make sense of the definition if they have been exposed only to the messages that we found in the *Precalculus* and *Calculus* text? If they think back to their precalculus experience, they may remember that informal definitions were not appropriate for proving rigorous claims. If they think back to their calculus experience, they may remember that one doesn't really have to make sense of definitions to answer questions. In both cases, they would be poorly prepared to answer the questions in the *Analysis* text.

<u>How are students supposed to use the definition?</u>

1. Questions deal with entailments of continuity

There are 74 questions having to do with continuity in *Analysis*, 26 of which are in the chapter on continuity, and the rest of which are in subsequent chapters.

There are no identification problems (e.g. show this function is continuous). Most of the questions include continuity in the hypothesis (If f is continuous and so-and-so, show f is such-and-such) or in the conclusion (Given a function which has such-and-such behavior, show that it is continuous.)

The nature of the questions is aimed at finding what the assumption of continuity buys you, or what conditions will yield continuity. In these questions, the definition is essential and the statement of the definition seems important. (Though it may not be clear to students that the particular definition given is not the only possibility for a definition.)

2. *Solutions require both syntax and semantics*

One example of a problem is, in content, the same as problem 60 from *Calculus*. Here it is stated:

> Let $I = [0,1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

Comparing this statement with the statement of problem 60 above, we can get a sense for the type of reasoning expected of the students in *Analysis*. There is no mention here of fixed points, only the condition which defines it. So students are not given the semantic interpretation of $f(x) = x$ which might help them make sense of this problem situation.

Students are also not told to use the Intermediate Value Theorem. So students must figure out what entailments of continuity are useful for this problem.

Notice that students need some semantic understanding of this problem situation to be able to solve it. Here, a syntactic understanding of continuity, like the *Calculus* text requires, isn't sufficient. Students need some semantic understanding to reason. They need to go back and forth between the syntax and the semantics.

3. *Continuity used throughout the book*

The fact that there are many problems on continuity in subsequent chapters of the book sends the message that this is an important topic. This is an important departure from both the *Precalculus* and *Calculus* treatments, where there is only one section on continuity. But it is disturbing that students would have to wait (and few do) until analysis to see the motivation for the topic. And because in earlier levels students can answer questions without paying much attention to the content of the questions, they may treat the *Analysis* questions in the same way.

What messages are sent at this level?
Now we are in a position to compare messages from all three levels. We see that in *Analysis*, the formal definition is not connected to informal characterizations. This differs from *Precalculus* where the informal characterization was the only one given and from *Calculus* where the informal characterization was given but not used. In *Analysis*, continuity is treated in a general setting in a way that students may not recognize the usefulness of concrete examples of real valued functions.

Second, we see again a problem regarding the use of informal and formal reasoning. In *Precalculus*, the problem was that the two were confounded. In *Calculus*, the problem was that the latter was treated exclusively at the expense of the former. And in *Analysis*, the problem is that based on students' previous experience, it may not be clear to them that problems require both formal and informal reasoning. No methods are given here. It appears the goal here is to satisfy a definition, like it was in *Calculus*. But in *Calculus* the meaning of the definition was irrelevant. Here it is essential. Here both an informal and formal understanding is important.

In short, in *Analysis*, both syntax and semantics are important, not only for reasoning but for learning new truths. *Analysis* authors assume students will learn content from the problems they are asked to solve. If students are given many problems in prior courses where the semantical content of the questions is unimportant, they may miss this important point later on.

SUMMARY

We have now seen more precisely the entailments of the fact that there are different epistemological assumptions at each of these three curricular levels. We find at each of the three levels, a different definition of continuity with a different purpose to be used by students in three different ways. The message sent at each level is different, and requires a different type of orientation towards the mathematics.

Below I have summarized the text messages of *Precalculus*, *Calculus*, and *Analysis* treatments of continuity.

Precalculus
Type of definition: Informal
Use: Classify
Characterization: Not clear (Questions seem to require only semantics, but text attempts to use syntax)

Calculus
Type of definition: Formal
Use: Satisfy a definition
Characterization: Requires only syntax

Analysis
Type of definition: Formal
Use: Find entailments
Characterization: Requires both syntax and semantics

Main claim of paper:
Given that

(1) there are different epistemological messages at different levels, and

(2) that students have notorious difficulty making transitions from high school to lower division and from lower division to upper division mathematics,

we should

(1) make explicit the epistemological messages at each level, and

(2) think about how to build on students' understandings to help them acquire an appropriate orientation to mathematics at each level.

DISCUSSION

The aim of this study was to look closely at messages sent by representative textbooks in courses that span from high school to college level mathematics. I made several methodological choices to help narrow the focus of the study, both in terms of what I have chosen as data and how I have chosen to view that data. It is now time to step back, in light of the claims of this paper, to see how reasonable those choices were, both to analyze the scope of the claims and to point for directions for future study.

Audience

One issue that may be important for interpreting this study is the audience for each text. Perhaps part of the reason for confusing messages at the different levels is that, at least in precalculus and calculus, the texts are trying to serve groups of students with very different mathematical needs. So one reading of this study is that we might, as Wu (Wu, in press). suggests, consider different types of classes for students with different mathematical needs. I do not want to enter the very heated and complicated debate about tracking, but I think it is harmless to point out that in order to find out how to create courses to meet the needs of all students, we must at least be aware of the kinds of messages being sent to students at each level.

Effect of teachers on students' beliefs

Another issue to keep in mind in interpreting this study is the role of the teacher. While textbooks are likely to have an impact on students' beliefs, teachers also play a role in either complimenting or contradicting text messages. Given that role, it may be important to keep in mind that precalculus, calculus, and analysis courses are taught by teachers with different levels of mathematical competency. Most high school teachers teaching precalculus or calculus often teach at the upper end of their mathematical competency, and college teachers teaching those courses or analysis are teaching nearer the bottom. This may have an impact on the assumptions brought out at different levels. However, it isn't clear to what extent a teacher's knowledge of mathematics can penetrate the epistemological assumptions of a curriculum.

Choice of texts

In this study I chose texts that were representative of mainstream texts at each level. As a result, I have passed over texts that are much less problematic. There are calculus texts, for instance, from both before and after the calculus reform movement, that do a nice job of motivating the notion of continuity and have a few more interesting questions (Hughes-Hallett, et al., 1998; Shenk, 1979).

In addition to textbooks, there are also articles suggesting better ways to teach continuity. For instance, Nadler (Nadler, 1994) is concerned with "what appear to be inappropriate pedagogical considerations regarding the introduction of concepts and their accompanying definitions in textbooks." To illustrate, he uses the example of continuity and provides a new definition, which he claims to better capture the fundamental idea behind the notion while providing some rigor.

However, since these treatments of continuity are not (yet) mainstream, the issues raised in this study are still relevant.

Next step

Textbooks provide one perspective on the issue of students' epistemological beliefs. But in order to determine whether I have correctly identified the messages sent at each level, I must also find out if students actually pick up on these messages. So my next step is to conduct empirical studies (videotape studies of classes and interviews with teachers and students) to attempt to triangulate my claims.

REFERENCES

D. Hughes-Hallett, et al, (1998). *Calculus* (Second ed.). New York: John Wiley.

S. Jacobsson, Wallin, H., & Wiklund, S. (1995). *Mathematics, Program N, Course C and D (in Swedish)* . Stockholm: Liber.

S. Nadler. (1994). Definitions and their Motivation: Continuity and Limits. *PRIMUS, 4* (3), 244-8.

A. H. Schoenfeld. (1989). Explorations of Students' Mathematical Beliefs and Behavior. *Journal for Research in Mathematics Education, 20* (4), 338-55.

M. Schommer, et al. (1992). Epistemological Beliefs and Mathematical Text Comprehension: Believing It Is Simple Does Not Make It So. *Journal of Educational Psychology, 84* (4), 435-43.

A. Shenk. (1979). *Calculus and Analytic Geometry* (Second ed.). Santa Monica, CA: Goodyear.

D. Tall. (1992). The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity, and Proof. In D. A. Grouws (Ed.), *Handbook of Research in Mathematics Teaching and Learning* . New York: Simon & Schuster Macmillan, 495-511.

D. Tall, & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity. *Educational Studies in Mathematics, 12* , 151-169.

H. Wu. (in press). On the education of math majors. In E. Gavosto, S. G. Krantz, & W. G. McCallum (Ed.), *Issues in Contemporary Mathematics Instruction* Cambridge: Cambridge University Press.